L34 April 14 Circle

Thursday, April 9, 2015 8:05 PM

To discuss the situation of S^1, it is easier to draw pictures in $\mathbb{R}^2\setminus\{0\}$. We first establish the relation between them.

Theorem If $A \subset X$ is a deformation retract then $\pi_1(A, x_0) = \pi_1(X, x_0)$ for $x_0 \in A$

Retract
$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, dashed, bend right, "id_A"'] & X \arrow[r, "r"] & A \end{tikzcd} \qquad r|_A = id_A$$

Deformation retract
$$\begin{tikzcd} X \arrow[r, "r"] \arrow[rr, dashed, bend right, "\simeq id_X"'] & A \arrow[r, hook, "i"] & X \end{tikzcd}$$

$$\left.\begin{array}{l} r \circ i = id_A \\ i \circ r \simeq id_X \end{array}\right\} \quad X \simeq A \text{ homotopy equivalent}$$

Fact $S^1 \simeq \mathbb{R}^2\setminus\{0\} = S^1 \times \mathbb{R}$

Thus, $\pi_1(S^1) = \pi_1(\mathbb{R}^2\setminus\{0\})$

In the following, we will show
$$\pi_1(\mathbb{R}^2\setminus\{0\}) = (\mathbb{Z}, +)$$

L34 p2

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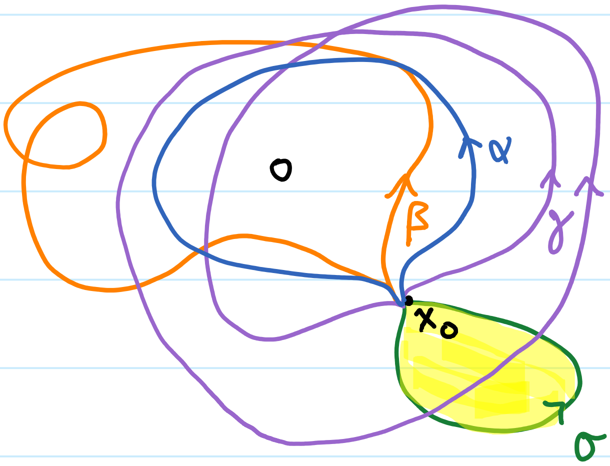
Examples. On $\mathbb{R}^2 \setminus \{(0,0)\}$



Intuitively,

$[\sigma] \longmapsto 0$

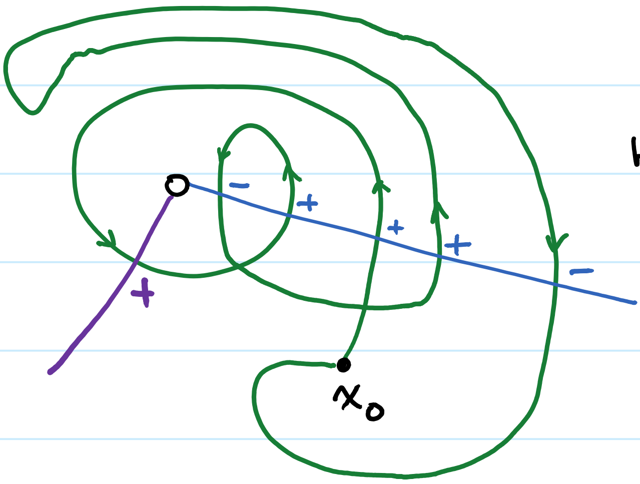
$[\alpha] = [\beta] \longmapsto 1$

$[\alpha]^{-1} = [\bar{\alpha}] \longmapsto -1$

$[\gamma] \longmapsto 2$

Expect that $\pi_1(\mathbb{R}^2 \setminus \{0\}) = (\mathbb{Z}, +)$

Combinatorial Calculation



$\longmapsto 3 - 2 = 1$

$1 = 1$

Need independent of choice of line

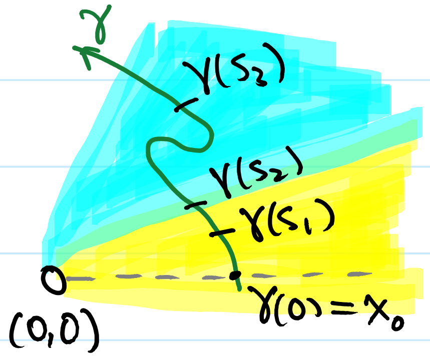
Complex Analysis $f(z) = z$

$$\Gamma \longmapsto \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-0}\, dz$$

$$= n(\Gamma, 0)$$

winding number at 0

Lifting (related to covering space theory)



Choose an open cover of $\mathbb{R}^2\setminus\{(0,0)\}$ by

$U_{a,b} = \{a < \theta < b\}$

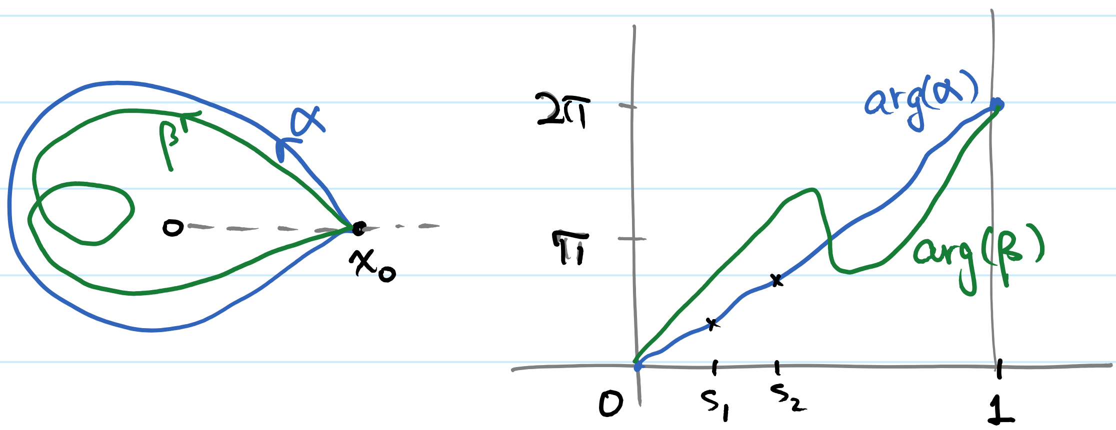
$0 < b-a < \pi/4 < 2\pi$

$\{\gamma^{-1}(U_{a,b})\}$ defines an open cover by intervals for $[0,1]$; its finite subcover gives a partition

$$0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$$

$$\gamma|_{[s_k, s_{k+1}]} : [s_k, s_{k+1}] \longrightarrow \text{some } U_{a,b}$$

A continuous choice of argument is equivalent to a function $[0,1] \longrightarrow \mathbb{R}$



$$\gamma \longmapsto \frac{1}{2\pi}\left(\arg\gamma(1) - \arg\gamma(0)\right)$$

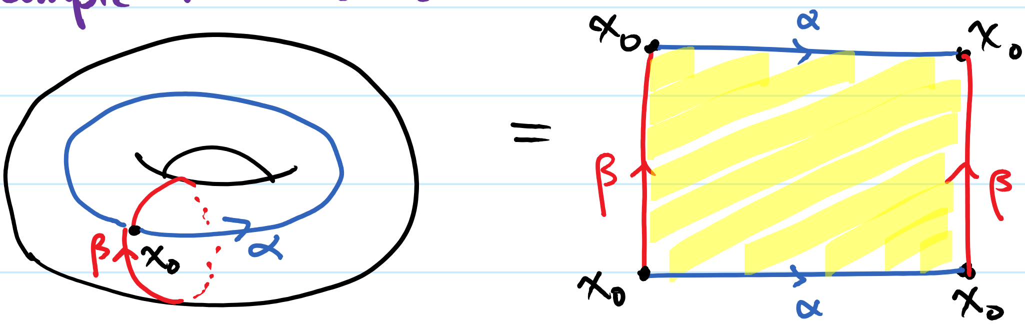
multiple of 2π

Example 1: $\pi_1(\text{contractible}) = 1$

Example 2: $\pi_1(S^n) = 1$, $n \geq 2$

Example 3: $\pi_1(S^1) = \pi_1(\mathbb{R}^2 \setminus \{0\}) = (\mathbb{Z}, +)$

Example 4 Torus



$$[\alpha] \longmapsto (1, 0)$$

$$[\beta] \longmapsto (0, 1)$$

Moreover, the loop $\alpha * \beta * \bar{\alpha} * \bar{\beta} \simeq c$,

$$\therefore [\alpha][\beta] = [\beta][\alpha]$$

Every loop based at x_0 on the torus

$\simeq$ a product of α and β

$= [\alpha]^m \cdot [\beta]^n \longmapsto (m, n)$

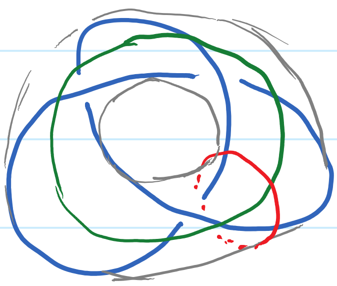
$$\therefore \pi_1(\text{torus}) = \mathbb{Z} \oplus \mathbb{Z}$$

Theorem. Let X, Y be path connected and $x_0 \in X$, $y_0 \in Y$. Then

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Torus $= S^1 \times S^1$ and $(\mathbb{Z}, +) \times (\mathbb{Z}, +) = \mathbb{Z} \oplus \mathbb{Z}$

Digression. Torus knot



Trefoil $\simeq$ (2,3)-loop

Example 4 Surface of genus g